SYMMETRY DIFFEOMORPHISM GROUP
OF A MANIFOLD OF NONPOSITIVE CURVATURE

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ABSTRACT. Let $\tilde{M}$ denote a complete simply connected manifold of nonpositive sectional curvature. For each point $p \in \tilde{M}$ let s_p denote the diffeomorphism of $\tilde{M}$ that fixes p and reverses all geodesics through p. The symmetry diffeomorphism group G^* generated by all diffeomorphisms $\{s_p : p \in \tilde{M}\}$ extends naturally to group of homeomorphisms of the boundary sphere $\tilde{M}(\infty)$. A subset X of $\tilde{M}(\infty)$ is called *involutive* if it is invariant under G^*.

THEOREM. *Let* $X \subseteq \tilde{M}(\infty)$ *be a proper, closed involutive subset. For each point* $p \in \tilde{M}$ *let* $N(p)$ *denote the linear span in* $T_p\tilde{M}$ *of those vectors at* p *that are tangent to a geodesic* γ *whose asymptotic equivalence class* $\gamma(\infty)$ *belongs to* X. *If* $N(p)$ *is a proper subspace of* $T_p\tilde{M}$ *for some point* $p \in \tilde{M}$, *then* $\tilde{M}$ *splits as a Riemannian product* $\tilde{M}_1 \times \tilde{M}_2$ *such that* N *is the distribution of* $\tilde{M}$ *induced by* $\tilde{M}_1$.

This result has several applications that include new results as well as great simplifications in the proofs of some known results. In a sequel to this paper it is shown that if $\tilde{M}$ is irreducible and $\tilde{M}(\infty)$ admits a proper, closed involutive subset X, then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$.

Introduction. Let $\tilde{M}$ denote a complete, simply connected manifold of nonpositive sectional curvature. It is well known that if p, q are any two distinct points of $\tilde{M}$, then there is a unique geodesic γ_{pq} that joins p to q. Hence for each point p of $\tilde{M}$ we obtain a symmetry diffeomorphism $s_p : \tilde{M} \to \tilde{M}$ given by $s_p\gamma(t) = \gamma(-t)$ for all geodesics γ of $\tilde{M}$ with $\gamma(0) = p$ and for all $t \in \mathbf{R}$. We define the symmetry diffeomorphism group G^* to be the group of diffeomorphisms of $\tilde{M}$ generated by the geodesic symmetries s_p, $p \in \tilde{M}$. The manifold $\tilde{M}$ is said to be a symmetric space if $G^* \subseteq I(\tilde{M})$, the isometry group of $\tilde{M}$, and $\tilde{M}$ is said to be *symmetric of noncompact type* if $\tilde{M}$ is a symmetric space and has a trivial Euclidean de Rham factor.

If $\tilde{M}(\infty)$ denotes the boundary sphere of an arbitrary manifold $\tilde{M}$ (see §1 for definitions and details), then the group G^* acts on $\tilde{M}(\infty)$ by homeomorphisms in a natural way. A subset $X \subseteq \tilde{M}(\infty)$ is said to be *involutive* if X is invariant under the group G^*. In this paper and its sequel [**E8**] we investigate the consequences when $\tilde{M}(\infty)$ admits a proper involutive subset X that is closed in the sphere topology of $\tilde{M}(\infty)$. The results fall into two categories: splitting theorems for $\tilde{M}$ and characterizations of symmetric spaces of noncompact type and rank at least two.

Note that a priori the group G^* need not contain any isometries of $\tilde{M}$ other than the identity.

Let $X \subseteq \tilde{M}(\infty)$ be a proper, closed involutive subset. For each point p of $\tilde{M}$ let $\mathcal{N}(p)$ denote the linear span in $T_p\tilde{M}$ of $\{V(p,x) : x \in X\}$, where $V(p,x)$ denotes the initial velocity of the unique geodesic γ_{px} that starts at p and belongs to x (compare §1). The main result of this paper, Theorem A of §3, is that if $\mathcal{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for some point p of $\tilde{M}$, then $\tilde{M}$ splits as a nontrivial Riemannian product $\tilde{M}_1 \times \tilde{M}_2$ such that the foliation of $\tilde{M}$ induced by $\tilde{M}_1$ is precisely the distribution $\mathcal{N}$. This result leads to simplified proofs of many known splitting theorems (see §§5 and 6), and in addition gives a cohesiveness to these theorems by putting them in a unified context.

The idea of an involutive subset $X \subseteq \tilde{M}(\infty)$ plays an important role in the proof of the main result of [**E6**], which is a special case of the Gromov Rigidity Theorem [**BGS**, §§14 and 15]; see especially Proposition 4.3 of [**E6**]. The symmetry diffeomorphism group G^* acting on $\tilde{M}(\infty)$ also plays a role in [**BS**]. The statement of Theorem A has its origin in the discussion of §5 of [**E6**], but the actual proof of Theorem A is very similar to the proof of a splitting theorem from [**BGS**, p. 222], which in fact is a special case of Theorem A.

In the sequel [**E8**] to this paper we show that if $\tilde{M}$ is irreducible in the sense of de Rham and admits a proper, closed involutive subset $X \subseteq \tilde{M}(\infty)$, then $\tilde{M}$ is isometric to a symmetric space of noncompact type and rank $k \geq 2$. This result yields as corollaries several characterizations of symmetric spaces of noncompact type and rank at least two, and in particular leads to simplified proofs of the Gromov Rigidity Theorem and the Rank Rigidity Theorem of Ballmann [**B**] and Burns-Spatzier [**BS**].

1. Preliminaries. In this section we summarize briefly some facts that will be useful later. For general references see [**BO, EO** and **BBE**].

(1.1) *Notation.* $\tilde{M}$ will always denote a complete, simply connected Riemannian manifold with nonpositive sectional curvature. All geodesics of $\tilde{M}$ will be assumed to have unit speed. $I(\tilde{M})$ denotes the group of isometries of $\tilde{M}$, and $I_0(\tilde{M})$ denotes the connected component of $I(\tilde{M})$ that contains the identity. $S\tilde{M}$ denotes the unit tangent bundle of $\tilde{M}$. Given $v \in S\tilde{M}$, the geodesic with initial velocity v is denoted by γ_v.

Geodesics γ, σ of $\tilde{M}$ are *asymptotic* if $d(\gamma t, \sigma t) \leq c$ for all $t \geq 0$ and some $c > 0$. $\tilde{M}(\infty)$ will denote the set of equivalence classes of asymptotic geodesics of $\tilde{M}$. Equipped with a natural sphere (cone) topology $\tilde{M}(\infty)$ is homeomorphic to an $(n-1)$ sphere, where $n = \dim \tilde{M}$. For a geodesic γ of $\tilde{M}$ we let $\gamma(\infty), \gamma(-\infty)$ denote the asymptotic equivalence classes of γ and $\gamma^{-1} : t \to \gamma(-t)$. Given points $p \in \tilde{M}$ and $x \in \tilde{M}(\infty)$ there exists a unique geodesic γ_{px} such that $\gamma_{px}(0) = p$ and $\gamma_{px}(\infty) = x$. $V(p,x)$ will denote the unit vector $\gamma'_{px}(0)$. The angle subtended at a point p in $\tilde{M}$ by points x, y in $\tilde{M}(\infty)$ is denoted by $\angle_p(x,y)$ and is defined to be the angle between $V(p,x)$ and $V(p,y)$.

(1.2) *Busemann functions and horospheres.* Given a point $x \in \tilde{M}(\infty)$ and a point $p \in \tilde{M}$ one defines a Busemann function f_x at x by

$$f_x(q) = \lim_{t \to +\infty} d(q, \gamma_{px}t) - t.$$

For different points p_1, p_2 in $\tilde{M}$ the corresponding Busemann functions at a fixed point x in $\tilde{M}(\infty)$ differ in $\tilde{M}$ by a constant. For any $x \in \tilde{M}(\infty)$ a Busemann function f_x at x is C^2 and convex, and $\operatorname{grad} f_x(p) = -V(p, x)$. Hence $|f_x(p) - f_x(q)| \leq d(p, q)$ for any $x \in \tilde{M}(\infty)$ and any points $p, q \in \tilde{M}$.

If f_x is a Busemann function at $x \in \tilde{M}(\infty)$ and if $p \in \tilde{M}$ is any point, then one defines $H(p, x) = \{q \in \tilde{M} : f_x(q) = f_x(p)\}$ and $B(p, x) = \{q \in \tilde{M} : f_x(q) \leq f_x(p)\}$. The sets $H(p, x)$ and $B(p, x)$ are called the *horosphere* and *horoball* through p determined by x. The set $B(p, x)$ is a proper closed convex subset of $\tilde{M}$ by the convexity of f_x.

(1.3) *Riemannian splittings of $\tilde{M}$.* Let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ be a Riemannian product manifold. This splitting of $\tilde{M}$ is said to be *invariant* under a group $\Gamma \subseteq I(\tilde{M})$ if for each $\varphi \in \Gamma$ and each $p \in \tilde{M}$ we have $d\varphi[\mathscr{N}_i(p)] = \mathscr{N}_i(\varphi p)$ for $i = 1, 2$, where $\mathscr{N}_i$ is the foliation of $\tilde{M}$ induced by $\tilde{M}_i$. If a group $\Gamma \subseteq I(\tilde{M})$ leaves invariant a splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$, then it follows that every element φ of Γ can be written $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i \in I(\tilde{M}_i)$ for $i = 1, 2$. In this case one has projection homomorphisms $p_i \colon \Gamma \to I(\tilde{M}_i)$ given by $p_i(\varphi) = \varphi_i$ for $i = 1, 2$. If $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ is a Riemannian product manifold with a trivial Euclidean de Rham factor, then every group $\Gamma \subseteq I(\tilde{M})$ admits a finite index subgroup Γ^* that leaves the splitting invariant.

(1.4) *Clifford translations.* For any isometry φ of $\tilde{M}$ one defines the displacement function $d_\varphi \colon \tilde{M} \to \mathbf{R}$ given by $d_\varphi(p) = d(p, \varphi p)$. The function d_φ is convex and its minimum locus, if nonempty, is the union of all geodesics in $\tilde{M}$ that are translated by φ.

If d_φ is constant in $\tilde{M}$, then the isometry φ is called a *Clifford translation* of $\tilde{M}$. If d_φ is bounded above in $\tilde{M}$, then φ is a Clifford translation of $\tilde{M}$ by elementary convexity properties. Let $C(\tilde{M})$ denote the subgroup of $I(\tilde{M})$ consisting of all Clifford translations.

Write $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1$, where $\tilde{M}_0$ is the Euclidean de Rham factor of $\tilde{M}$ and $\tilde{M}_1$ is the product of all irreducible non-Euclidean de Rham factors of $\tilde{M}$. A result of [**W1**] states that $C(\tilde{M}) = \{\varphi \in I(\tilde{M}) : \varphi = (\varphi_0, \mathrm{id})$, where φ_0 is a translation of $\tilde{M}_0\}$. In particular $C(\tilde{M})$ is a normal abelian subgroup of $I(\tilde{M})$.

(1.5) *Limit sets.* If $\Gamma \subseteq I(\tilde{M})$ is any subgroup and $p \in \tilde{M}$ is any point, then one defines the *limit set* $L(\Gamma)$ to be the set of accumulation points in $\tilde{M}(\infty)$ of the orbit $\Gamma(p)$. The set $L(\Gamma)$ is a closed subset of $\tilde{M}(\infty)$ in the sphere topology. $L(\Gamma)$ does not depend on the point $p \in \tilde{M}$, and hence $L(\Gamma)$ is invariant under all isometries of $\tilde{M}$ that normalize Γ.

For later use we need the following.

PROPOSITION. *Let $\Gamma \subseteq I(\tilde{M})$ be a subgroup such that $L(\Gamma) = \tilde{M}(\infty)$, and let N be a nonidentity normal subgroup of Γ. Then $L(N)$ is nonempty.*

PROOF. Let N, Γ be as in the statement of the proposition. If $L(N)$ is empty, then the orbit $N(p)$ is bounded in $\tilde{M}$ for every $p \in \tilde{M}$ and hence N has a fixed point in $\tilde{M}$ by the Cartan fixed point theorem. The set of points in $\tilde{M}$ fixed by N is closed, convex and invariant under Γ. Since $L(\Gamma) = \tilde{M}(\infty)$ it follows that N fixes every point of $\tilde{M}$, which contradicts the hypothesis that $N \neq \{1\}$. $\quad\square$

(1.6) *Duality.* Let $\Gamma \subseteq I(\tilde{M})$ be any group. Two points x, y in $\tilde{M}(\infty)$, not necessarily distinct, are said to be Γ-*dual* if there exists a sequence $\{\varphi_n\} \subseteq \Gamma$ such that $\varphi_n(p) \to x$ and $\varphi_n^{-1}(p) \to y$ as $n \to \infty$ for any point p of $\tilde{M}$. It is routine to verify that the set of points in $\tilde{M}(\infty)$ that are Γ-dual to a given point $x \in \tilde{M}(\infty)$ is closed in $\tilde{M}(\infty)$ and invariant under Γ. The next result, whose proof is contained in the proof of Lemma 2.4a of [**CE**], will be useful later.

PROPOSITION 1.6a. *Let $1 \neq \Gamma \subseteq I(\tilde{M})$ be any group, and let x, z be points in $\tilde{M}(\infty)$ that are Γ-dual. If $p \in \tilde{M}$ is arbitrary and $y = \gamma_{px}(-\infty) \in \tilde{M}(\infty)$, then $z \in \overline{\Gamma(y)}$.*

A group $\Gamma \subseteq I(\tilde{M})$ is said to satisfy the *duality condition* if $\gamma(\infty)$ and $\gamma(-\infty)$ are Γ-dual for any geodesic γ of $\tilde{M}$. If $\Gamma \subseteq I(\tilde{M})$ is a discrete group such that the quotient space $\tilde{M}/\Gamma$ is a smooth manifold of finite Riemannian volume, then Γ satisfies the duality condition by Proposition 3.7 of [**E2**] and the Poincaré recurrence lemma.

PROPOSITION 1.6b. *Let $\Gamma \subseteq I(\tilde{M})$ satisfy the duality condition. Then Γ has the following properties*:

(1) *If Γ^* is a subgroup of Γ with finite index, then Γ^* satisfies the duality condition.*

(2) *If Γ leaves invariant a Riemannian splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$, then $p_i(\Gamma)$ satisfies the duality condition in $\tilde{M}_i$ for $i = 1, 2$, where $p_i \colon \Gamma \to I(\tilde{M}_i)$ denotes the projection homomorphism.*

(3) *If $x \in \tilde{M}(\infty)$ is arbitrary, then $\overline{\Gamma(y)} = \overline{\Gamma(x)}$ for all $y \in \overline{\Gamma(x)}$.*

PROOF. Property (1) is proved in Appendix I of [**E3**] while property (3) is proved in Lemma 2.9 of [**BBE**]. Property (2) follows from the definition of the duality condition and geometric properties of Riemannian product manifolds. □

2. The symmetry diffeomorphism group. Let $\tilde{M}$ be an arbitrary complete, simply connected Riemannian manifold with nonpositive sectional curvature. For each point p in $\tilde{M}$ one defines the geodesic symmetry $s_p \colon \tilde{M} \to \tilde{M}$ given by

$$s_p(\gamma(t)) = \gamma(-t)$$

for all geodesics γ of $\tilde{M}$ with $\gamma(0) = p$ and for all $t \in \mathbf{R}$. The map s_p fixes p and is a diffeomorphism of $\tilde{M}$. We define

(2.1) $G^* =$ the subgroup of diffeomorphisms of $\tilde{M}$
                      generated by the geodesic symmetries $\{s_p \colon p \in \tilde{M}\}$.

(2.2) $G_e^* =$ the subgroup of G^* consisting of
                       elements that are the product of an even
                       number of geodesic symmetries.

The group G^* is called the symmetry diffeomorphism group of $\tilde{M}$. The subgroup G_e^*, which has index 2 and hence is normal in G^*, is called the group of *even* symmetry diffeomorphisms (compare [**L**, p. 64]).

(2.3) G^* *acting on* $\tilde{M}(\infty)$. For each point $p \in \tilde{M}$ we define $s_p \colon \tilde{M}(\infty) \to \tilde{M}(\infty)$ by

$$s_p(x) = \gamma_{px}(-\infty)$$

for every $x \in \tilde{M}(\infty)$. The group G^* now acts on $\tilde{M}(\infty)$ by homeomorphisms. Note that $\varphi \circ s_p \circ \varphi^{-1} = s_{\varphi(p)}$ for all $\varphi \in I(\tilde{M})$ and all $p \in \tilde{M}$ so that $I(\tilde{M})$ normalizes both G^* and G_e^*, regarded as groups acting on $\tilde{M}$ by diffeomorphisms or on $\tilde{M}(\infty)$ by homeomorphisms.

(2.4) *Involutive subsets of* $\tilde{M}(\infty)$.

DEFINITION. A subset $X \subseteq \tilde{M}(\infty)$ is called *involutive* if X is invariant under the symmetry diffeomorphism group G^*.

REMARKS. (a) If $X \subseteq \tilde{M}(\infty)$ is involutive, then the closure of X in $\tilde{M}(\infty)$ is also involutive.

(b) If $X \subseteq \tilde{M}(\infty)$ is invariant under G_e^*, the subgroup of even symmetry diffeomorphisms, then $X \cup s_p(X)$ and $X \cap s_p(X)$ are involutive subsets for every point p in $\tilde{M}$. This follows from the fact that G_e^* is a normal subgroup of G^* and hence $s_p(X)$ is G_e^*-invariant for every $p \in \tilde{M}$.

Examples of involutive subsets. (a) Let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ be a Riemannian product manifold. Given any numbers α, β with $0 \le \alpha, \beta \le 1$ and $\alpha^2 + \beta^2 = 1$ we define

$$\tilde{M}_{\alpha\beta}(\infty) = \{\gamma(\infty) \colon \gamma(t) = (\gamma_1(\alpha t), \gamma_2(\beta t)),$$

$$\text{where } \gamma_i \text{ is a unit speed geodesic in } M_i \text{ for } i = 1, 2\}.$$

The set $\tilde{M}_{\alpha\beta}(\infty)$ is closed and involutive for each choice of α, β. Note that $\tilde{M}_{\alpha\beta}(\infty) = \tilde{M}_1(\infty)$ if $\alpha = 1$ and $\beta = 0$ and equals $\tilde{M}_2(\infty)$ if $\alpha = 0$ and $\beta = 1$.

(b) Let $\tilde{M}$ be a symmetric space of noncompact type. If $X \subseteq \tilde{M}(\infty)$ is invariant under the isometry group $I(\tilde{M})$, then X is involutive since $G^* \subseteq I(\tilde{M})$. If $\tilde{M}$ has rank $k \ge 2$, then the closure in $\tilde{M}(\infty)$ of any orbit of $I(\tilde{M})$ is a proper subset of $\tilde{M}(\infty)$. Moreover, the set $X \subseteq \tilde{M}(\infty)$ consisting of singular points (compare §3 of [**E6**]) is proper, closed and invariant under $I(\tilde{M})$.

(2.5) *Existence criteria for involutive subsets.*

PROPOSITION 2.5a. *Let* $\Gamma \subseteq I(\tilde{M})$ *satisfy the duality condition, and let* $X \subseteq \tilde{M}(\infty)$ *be closed and* Γ-*invariant. Then* $X \cup s_p(X)$ *and* $X \cap s_p(X)$ *are closed involutive subsets of* $\tilde{M}(\infty)$ *for any point* p *of* $\tilde{M}$.

PROOF. This is an immediate consequence of (b) in (3.2) above and the following.

LEMMA 2.5a. *Let* $\Gamma \subseteq I(\tilde{M})$ *satisfy the duality condition, and let* $X \subseteq \tilde{M}(\infty)$ *be closed and* Γ-*invariant. Then* X *is* G_e^*-*invariant.*

PROOF OF THE LEMMA. It suffices to show that $s_p s_q(x) \in \overline{\Gamma(x)}$ for any points p, q in $\tilde{M}$ and $x \in X$. If $y = s_q(x)$ and $x^* = s_p(y) = s_p s_q(x)$, then y is Γ-dual to x^* and the assertion of the lemma now follows from Proposition 1.6a. $\square$

COROLLARY 2.5a. *Let* $\Gamma \subseteq I(\tilde{M})$ *satisfy the duality condition. If* $\tilde{M}(\infty)$ *admits a proper closed* Γ-*invariant subset, then* $\tilde{M}(\infty)$ *admits a proper closed involutive subset.*

PROOF. Let $X \subseteq \tilde{M}(\infty)$ be a proper closed Γ-invariant subset. The sets $X \cup s_p(X)$ and $X \cap s_p(X)$ are closed and involutive for any point p of $\tilde{M}$ by the proposition. If $X \cup s_p(X) = \tilde{M}(\infty)$ for some point p of $\tilde{M}$, then $X \cap s_p(X)$ is nonempty since $\tilde{M}(\infty)$ is connected. □

PROPOSITION 2.5b. *Let $\Gamma \subseteq \tilde{M}(\infty)$ satisfy the duality condition, and let $X \subseteq \tilde{M}(\infty)$ be closed and Γ-invariant. Assume moreover that for each point $x \in X$ there exists $y \in X$ that is Γ-dual to x. Then X is involutive.*

PROOF. Given $x \in X$ and $p \in \tilde{M}$ let $y = s_p(x) \in \tilde{M}(\infty)$. If $y^* \in X$ is Γ-dual to x, then by Proposition 1.6a we have $y^* \in \overline{\Gamma(y)}$, and hence $y \in \overline{\Gamma(y^*)} \subseteq X$ by (3) of Proposition 1.6b.

COROLLARY 2.5b. *Let $\Gamma \subseteq I(\tilde{M})$ satisfy the duality condition, and let $1 \neq \Gamma_0$ be a normal subgroup of Γ. Then $L(\Gamma_0)$, the limit set of Γ_0, is a closed, nonempty involutive subset of $\tilde{M}(\infty)$.*

PROOF. By the definition of $L(\Gamma_0)$, for every point $x \in L(\Gamma_0)$ there exists a point $y \in L(\Gamma_0)$ that is Γ_0-dual and hence Γ-dual to x. The assertion now follows from Proposition 2.5b and the proposition in (1.5). □

PROPOSITION 2.5c. *Let $\Gamma \subseteq I(\tilde{M})$ be a subgroup with $L(\Gamma) = \tilde{M}(\infty)$. Let $X \subseteq \tilde{M}(\infty)$ be a closed Γ-invariant subset such that for each point $x \in X$ there exists $p \in \tilde{M}$ such that $s_p(x) \in X$. Then X is involutive.*

PROOF. Given $x \in X$ and $q \in \tilde{M}$ let $y = s_q(x) \in \tilde{M}(\infty)$. Choose $\{\varphi_n\} \subseteq \Gamma$ so that $\varphi_n(q) \to y$ as $n \to \infty$ and let $\varphi_n^{-1}(q)$ converge to some point $z \in \tilde{M}(\infty)$ by passing to a subsequence if necessary. By Proposition 1.6a we see that $z \in \overline{\Gamma(x)} \subseteq X$, and by hypothesis there exists a point $p \in \tilde{M}$ such that $w = s_p(z) \in X$. Since y and z are Γ-dual it follows by Proposition 1.6a that $y \in \overline{\Gamma(w)} \subseteq X$. □

COROLLARY 2.5c. *Let $B \subseteq \tilde{M}$ be a complete, totally geodesic submanifold, and let $X = B(\infty) \subseteq \tilde{M}(\infty)$. Let $\Gamma_X = \{\varphi \in I(\tilde{M}): \varphi(X) \subseteq X\}$. If $L(\Gamma_X) = \tilde{M}(\infty)$, then X is involutive.*

From the corollary above and Theorem A of the next section we obtain an alternate proof of Proposition 2.2 of [E3]. However, the main results in this paper, Theorem A and its applications, are related to Corollary 2.5b.

3. A splitting theorem. In this section we prove the following result:

THEOREM A. *Let $\tilde{M}$ be an arbitrary complete, simply connected manifold of nonpositive sectional curvature, and let $X \subseteq \tilde{M}(\infty)$ be a closed involutive subset. For each point p in $\tilde{M}$ define*

$$B_p = closed\ convex\ hull\ of\ \{\gamma_{px}(\mathbf{R}): x \in X\},$$
$$\mathcal{N}(p) = subspace\ of\ T_p\tilde{M}\ spanned\ by\ \{\gamma'_{px}(0): x \in X\}.$$

Then
(1) B_p is a proper subset of $\tilde{M}$ for all points $p \in \tilde{M}$ if and only if $\mathcal{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for some point $p \in \tilde{M}$.

(2) *If $\mathcal{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for some point $p \in \tilde{M}$, then there exists a Riemannian splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ such that the distribution $p \to \mathcal{N}(p)$ is the foliation of $\tilde{M}$ induced by $\tilde{M}_1$ and the sets B_p are the maximal integral manifolds $\tilde{M}_1 \times \{p_2\}$ of $\mathcal{N}$, where $p = (p_1, p_2)$. Moreover, if $\phi \in I(\tilde{M})$ leaves X invariant, then ϕ leaves the splitting invariant.*

The proof requires three preliminary results.

LEMMA 1. *Let $X \subseteq \tilde{M}(\infty)$ be a closed involutive subset, and let $z \in \tilde{M}(\infty)$ and $p \in \tilde{M}$ be any points. Then the following statements are equivalent:*
(1) $\angle_p(z, x) \leq \pi/2$ *for all* $x \in X$.
(2) $\angle_p(z, x) \geq \pi/2$ *for all* $x \in X$.
(3) $\angle_p(z, x) = \pi/2$ *for all* $x \in X$.

PROOF OF LEMMA 1. Let $p \in \tilde{M}$ and $z \in \tilde{M}(\infty)$ be as above. Given a point $x \in X$ we define $y = S_p(x) = \gamma_{px}(-\infty)$. The point y lies in X since X is involutive. The equivalence of the statements above is now immediate since $\pi = \angle_p(z, x) + \angle_p(z, y)$. $\square$

LEMMA 2. *Let $\mathcal{N}(p)$ be a proper subspace of $T_p\tilde{M}$ for some point $p \in \tilde{M}$. Let $v \in \mathcal{N}(p)^\perp$ be any unit vector, and let $z = \gamma_v(\infty) \in \tilde{M}(\infty)$. Then $V(q, z) = \gamma'_{qz}(0) \in \mathcal{N}(q)^\perp$ for all $q \in \tilde{M}$.*

PROOF OF LEMMA 2. We show first that $\angle_{\gamma(t)}(z, x) = \pi/2$ for all $t \in \mathbf{R}$ and all $x \in X$, where $\gamma(t) = \gamma_v(t)$. It suffices to consider the case $t \geq 0$ since $\gamma(-t) = \sigma(t)$ has initial velocity $-v \in \mathcal{N}(p)^\perp$. Fix $x \in X$. The assertion is true for $t = 0$ since $\angle_p(z, x) = \angle(v, \gamma'_{px}(0)) = \pi/2$ by the fact that $v \in \mathcal{N}(p)^\perp$. If $\theta(t) = \angle_{\gamma(t)}(p, x)$ for $t > 0$, then $\theta(t) \leq \pi - \angle_p(z, x) = \pi/2$ since the sum of the interior angles of a geodesic triangle of $\tilde{M}$ is at most π, even when one of the "vertices" lies in $\tilde{M}(\infty)$. Hence $\angle_{\gamma(t)}(z, x) = \pi - \theta(t) \geq \pi/2$ for all $t > 0$. From Lemma 1 it now follows that $\angle_{\gamma(t)}(z, x) = \pi/2$ for all $t \in \mathbf{R}$ and all $x \in X$.

Now let $z^* = \gamma_v(-\infty) = \gamma_{-v}(\infty)$. We show that for every $x \in X$ the geodesic $\gamma_{px}(\mathbf{R})$ is contained in $H(p, z) \cap H(p, z^*)$, the intersection of the horospheres through p determined by z and z^* (see (1.2)). Fix $x \in X$. We showed above that $\angle_{\gamma_v(t)}(z, x) = \angle_{\gamma_v(t)}(z^*, x) = \pi/2$ for all $t \in \mathbf{R}$ and all $x \in X$. It follows from Sublemma 5.4a of [**E6**, p. 78] that $\gamma_v(\mathbf{R})$ and $\gamma_{px}[0, \infty]$ span a flat half plane in $\tilde{M}$ with boundary $\gamma_v(\mathbf{R})$. Hence $\gamma_{px}[0, \infty] \subseteq H(p, z) \cap H(p, z^*)$ by the Euclidean geometry of this flat half plane. Since $y = \gamma_{px}(-\infty) \in X$ it follows by the argument above that $\gamma_{py}[0, \infty] = \gamma_{px}[-\infty, 0] \subseteq H(p, z) \cap H(p, z^*)$. Hence $\gamma_{px}(\mathbf{R}) \subseteq H(p, z) \cap H(p, z^*)$.

We now complete the proof of Lemma 2. Let $q \in \tilde{M}$ and $x \in X$ be given. Let f_z be the Busemann function at z such that $f_z(q) = 0$. By the previous paragraph $f_z(\gamma_{px}t) \equiv c$ in $\mathbf{R}$. Hence for $t \geq 0$ we have

$$f_z(\gamma_{qx}t) = [f_z(\gamma_{qx}t) - f_z(\gamma_{px}t)] + f_z(\gamma_{px}t)$$
$$\leq c + d(\gamma_{qx}t, \gamma_{px}t) \leq c + d(p, q)$$

by (1.2) and the fact that γ_{px} and γ_{qx} are asymptotes. The convexity of f_z implies that $t \to f_z(\gamma_{qx}t)$ is nonincreasing in t and hence

$$\cos \angle_q(x, z) = -\langle \text{grad } f_z(q), \gamma'_{qx}(0)\rangle \geq 0$$

by (1.2). This proves that $\angle_q(z, x) \le \pi/2$ for all $q \in \tilde{M}$ and all $x \in X$ and equality holds by Lemma 1. It now follows from the definition of $\mathscr{N}(q)$ that $V(q, z) = \gamma'_{qz}(0) \in \mathscr{N}(q)^\perp$ for all $q \in \tilde{M}$. $\square$

LEMMA 3. *The following statements are equivalent:*
(1) B_p *is a proper subset of all points p of $\tilde{M}$.*
(2) B_p *is a proper subset for some point p of $\tilde{M}$.*
(3) $\mathscr{N}(p)$ *is a proper subspace of $T_p\tilde{M}$ for all points p of $\tilde{M}$.*
(4) $\mathscr{N}(p)$ *is a proper subspace of $T_p\tilde{M}$ for some point p of $\tilde{M}$.*

PROOF. The assertions (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious. We prove the assertions (4) $\Rightarrow$ (3), (3) $\Rightarrow$ (1), and (2) $\Rightarrow$ (4).

PROOF OF (4) $\Rightarrow$ (3). Suppose that $\mathscr{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for some point $p \in \tilde{M}$. Let $v \in \mathscr{N}(p)^\perp$ be any unit vector and let $z = \gamma_v(\infty)$. For any point $q \in \tilde{M}$ the unit vector $V(q, z) = \gamma'_{qz}(0)$ lies in $\mathscr{N}(q)^\perp$ by Lemma 2. $\square$

PROOF OF (3) $\Rightarrow$ (1). Assume that $\mathscr{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for all points $p \in \tilde{M}$, and let $p \in \tilde{M}$ be given. Let $v \in \mathscr{N}(p)^\perp$ be any unit vector and let $z = \gamma_v(\infty)$. In the proof of Lemma 2 we showed that for any $x \in X$ the geodesic $\gamma_{px}(\mathbf{R})$ lies in $H(p, z) \subseteq B(p, z) = \{q \in \tilde{M}: f_z(q) \le f_z(p)\}$, where f_z is any Busemann function at z. Since $B(p, z)$ is a proper, closed, convex subset of $\tilde{M}$ it follows from the definition of B_p that $B_p \subseteq B(p, z)$. Hence B_p is a proper subset of $\tilde{M}$. $\square$

PROOF OF (2) $\Rightarrow$ (4). Suppose that B_p is a proper subset for some point $p \in \tilde{M}$. Let r be any point of $\tilde{M} - B_p$, and let $q \in B_p$ be the unique point in B_p closest to r. Fix $x \in X$ and observe that $\gamma_{qx}[0, \infty] \subseteq B_p$; this follows from the fact that $\gamma_{px}(\mathbf{R}) \subseteq B_p$ and $\gamma_{qx}(t) = \lim_{n \to \infty} \gamma_n(t)$ for all $t \in \mathbf{R}$, where γ_n is the unique geodesic from q to $\gamma_{px}(n)$. If $z = \gamma_{qr}(\infty)$, then $\angle_q(z, x) = \angle_q(r, x) \ge \pi/2$ for otherwise the function $t \to d(r, \gamma_{qx}(t))$ would have a negative derivative at $t = 0$, contradicting the definition of q. This shows that $\angle_q(z, x) \ge \pi/2$ for all $x \in X$ and equality follows by Lemma 1. Hence if $v = \gamma'_{qz}(0)$, then $v \in \mathscr{N}(q)^\perp$ and $\mathscr{N}(q)$ is a proper subspace of $T_q\tilde{M}$. $\square$

PROOF OF THEOREM A. The first assertion is contained in Lemma 3. We now prove assertion (2), assuming that $\mathscr{N}(p)$ is a proper subspace of $T_p\tilde{M}$ for one and hence all points p of $\tilde{M}$ by Lemma 3.

We show first that the dimension of $\mathscr{N}(p)$ is constant in $\tilde{M}$. By the connectedness of $\tilde{M}$ it suffices to show that the dimension of $\mathscr{N}(p)$ is a locally constant function of p. We do this in two steps.
  (a) $\dim \mathscr{N}(q) \ge \dim \mathscr{N}(p)$ for all q near p.
  (b) $\dim \mathscr{N}(q) \le \dim \mathscr{N}(p)$ for all q near p.
To prove (a) we choose points $x_1, x_2, \ldots, x_N \in X$ so that the unit vectors $\{V(p, x_i): 1 \le i \le N\}$ are a basis of $\mathscr{N}(p)$. If $q \in \tilde{M}$ is sufficiently close to p, then the vectors $\{V(q, x_i): 1 \le i \le N\}$ remain linearly independent elements of $\mathscr{N}(q)$, which proves (a).

Now choose points $y_1, \ldots, y_r$ in $\tilde{M}(\infty)$ so that $\{V(p, y_j): 1 \le j \le r\}$ is a basis for $\mathscr{N}(p)^\perp$. If $q \in \tilde{M}$ is arbitrary, then $\{V(q, y_j) \in \mathscr{N}(q)^\perp$ for each $1 \le j \le r$

by Lemma 2, and if q is sufficiently close to p, then these vectors remain linearly independent, which proves (b).

A C^1 distribution $\mathscr{F}$ in $\tilde{M}$ is said to be *parallel* if for any C^1 curve $\gamma\colon [a,b] \to \tilde{M}$ we have

$$\mathscr{F}(\gamma(b)) \text{ is the parallel translate along } \gamma \text{ of } \mathscr{F}(\gamma(a)).$$

It is easy to see that a C^1 distribution $\mathscr{F}$ is parallel if and only if for any vector $v \in T_p\tilde{M}$ and any C^1 vector field Z tangent to $\mathscr{F}$ near p we have $\nabla_v Z \in \mathscr{F}(p)$.

Let $p \in \tilde{M}$ be given. By the discussion above we may choose a neighborhood 0 of p in $\tilde{M}$, points $x_1,\ldots,x_N$ in X and $y_1,\ldots,y_r$ in $\tilde{M}(\infty)$ such that $\{V(q,x_i)\colon 1 \le i \le N\}$ and $\{V(q,y_j)\colon 1 \le j \le r\}$ are bases for $\mathscr{N}(q)$ and $\mathscr{N}(q)^\perp$ for all $q \in 0$. Let V_i, W_j be those unit vector fields in $\tilde{M}$ defined by $V_i(q) = V(q,x_i)$ and $W_j(q) = V(q,y_j)$. The distribution $\mathscr{N}$ is C^1 since the vector fields $\{-V_i\}$ are the gradients of C^2 Busemann functions at x_i by the discussion in (1.2).

We show

$$(*) \qquad \nabla_{V_i}W_j \equiv 0 \quad \text{and} \quad \nabla_{W_j}V_i \equiv 0 \quad \text{in } \tilde{M}.$$

By definition the vector field V_i is tangent to $\mathscr{N}$ at all points of $\tilde{M}$ for every $1 \le i \le N$, and by Lemma 2 the vector field W_j is tangent to $\mathscr{N}^\perp$ at all points of $\tilde{M}$ for every $1 \le j \le r$. Fixing $1 \le i \le N$ and $1 \le j \le r$ we conclude by Sublemma 5.4a of [**E6**, p. 78] that for every point $q \in \tilde{M}$ the orthogonal geodesics $\gamma_{qx_i}(\mathbf{R})$ and $\gamma_{qy_j}[0,\infty]$ determine a flat half plane in $\tilde{M}$ whose boundary is $\gamma_{qx_i}(\mathbf{R})$. Hence the vector field $t \to W_j(\gamma t)$ is parallel on $\gamma(\mathbf{R}) = \gamma_{qx_i}(\mathbf{R})$, and it follows that $(\nabla_{V_i}W_j)(q) = 0$. Similarly $\nabla_{W_i}V_j \equiv 0$ in $\tilde{M}$.

To prove that $\mathscr{N}$ is a parallel distribution in $\tilde{M}$ it suffices by $(*)$ and the discussion above to show that $(\nabla_{V_i}V_j)(q) \in \mathscr{N}(q)$ for all $1 \le i,j \le N$ and all q in the given neighborhood 0 of p. For $1 \le k \le r$ and $1 \le i,j \le N$ we have

$$\langle \nabla_{V_i}V_j, W_k\rangle = V_i\langle V_j, W_k\rangle - \langle V_j, \nabla_{V_i}W_k\rangle = 0$$

by $(*)$ and the fact that V_i, V_j are orthogonal to W_k in $\tilde{M}$. Since $\{W_1(q),\ldots,W_r(q)\}$ is a basis for $\mathscr{N}(q)^\perp$ for all $q \in 0$ it follows that $\mathscr{N}$ is a parallel distribution in $\tilde{M}$.

Since $\mathscr{N}$ is parallel the de Rham theorem implies that $\tilde{M}$ splits as a Riemannian product $\tilde{M}_1 \times \tilde{M}_2$ such that the foliations of $\tilde{M}$ induced by $\tilde{M}_1, \tilde{M}_2$ are $\mathscr{N}, \mathscr{N}^\perp$ respectively [**KN**]. If F_p denotes the maximal integral manifold of $\mathscr{N}$ passing through p, then F_p is a complete totally geodesic submanifold of $\tilde{M}$. We show that $F_p = B_p$. By definition $\mathscr{N}(p) = T_pF_p$ and hence $\gamma_{px}(\mathbf{R}) \subseteq F_p$ for all $x \in X$ since $\gamma'_{px}(0) \in \mathscr{N}(p)$ for all $x \in X$. It follows that $B_p \subseteq F_p$ since F_p is a closed convex subset of $\tilde{M}$.

Suppose that B_p is a proper subset of F_p, and let r be any point in $F_p - B_p$. Let $q \in B_p$ be the unique point in B_p closest to r, and let $v = \gamma'_{qr}(0)$. The proof of assertion (2) $\Rightarrow$ (4) in Lemma 3 implies that $v \in \mathscr{N}(q)^\perp$, and $v \in T_qF_p$ since r lies in F_p. On the other hand $\mathscr{N}(q) = T_qF_p$ since F_p is complete and totally geodesic and $\mathscr{N}$ is parallel in $\tilde{M}$. This implies that v is a unit vector in $\mathscr{N}(q) \cap \mathscr{N}(q)^\perp$, a contradiction that implies $B_p = F_p$.

Finally, if $\varphi \in I(\tilde{M})$ leaves invariant the involutive subset $X \subseteq \tilde{M}(\infty)$, then $d\varphi \mathcal{N}(p) = \mathcal{N}(\varphi p)$ for every $p \in \tilde{M}$ by the definition of $\mathcal{N}$. Hence $d\varphi \mathcal{N}(p)^{\perp} = \mathcal{N}(\varphi p)^{\perp}$ for every $p \in \tilde{M}$. By definition (see (1.3)) it follows that φ leaves invariant the splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$, which completes the proof of Theorem A. □

4. Applications. In this section and the next we shall use Theorem A to derive some new results and to simplify the proofs of some known results. Our general theme is the study of groups $N \subseteq I(\tilde{M})$ whose normalizer in $I(\tilde{M})$ satisfies the duality condition. We begin with the following useful

LEMMA. *Let $N \subseteq I(\tilde{M})$ be a nonidentity subgroup whose normalizer Γ in $I(\tilde{M})$ satisfies the duality condition and whose centralizer Z in $I(\tilde{M})$ is not the identity. Then either*
(1) *$Z \subseteq C(\tilde{M})$, the subgroup of Clifford translations of $\tilde{M}$ or*
(2) *There exists a nontrivial Riemannian splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ that is invariant under Γ and Z and such that*
(a) *$N \subseteq I(\tilde{M}_1) \times \{1\}$,*
(b) *$p_1(Z) \subseteq C(\tilde{M}_1)$, where p_1 denotes the projection homeomorphism into $I(\tilde{M}_1)$.*

From the discussion in (1.4) we then obtain the following

COROLLARY. *Let $N \subseteq I(\tilde{M})$ be a nonidentity subgroup whose normalizer Γ in $I(\tilde{M})$ satisfies the duality condition. If $\tilde{M}$ is irreducible, then the centralizer Z of N in $I(\tilde{M})$ is the identity.*

PROOF OF THE LEMMA. By Corollary 2.5b the limit set $L(N)$ is a nonempty, closed, involutive subset of $\tilde{M}(\infty)$. Let B_p denote the closed convex hull of $\{\gamma_{px}(\mathbf{R}) : x \in L(N)\}$ for each point p of $\tilde{M}$. If $\varphi \in Z$, then $d_\varphi(\psi p) = d_\varphi(p)$ for all $\psi \in N$ since $\varphi\psi = \psi\varphi$, and by the convexity of d_φ it follows that $d_\varphi \leq d_\varphi(p)$ on $\gamma_{px}[0, \infty)$ for every $x \in L(N)$. Since $L(N)$ is involutive it follows that $d_\varphi \leq d_\varphi(p)$ on $\gamma_{px}(\mathbf{R})$ for every $x \in L(N)$ and hence $d_\varphi \leq d_\varphi(p)$ on B_p. If $B_p = \tilde{M}$ for some point p of $\tilde{M}$, then $Z \subseteq C(\tilde{M})$ by the discussion above in (1.4).

Suppose now that $Z \not\subseteq C(\tilde{M})$ and B_p is a proper subset of $\tilde{M}$ for every point p of $\tilde{M}$. By Theorem A in the previous section there exists a Riemannian splitting $\tilde{M} = M_1^* \times M_2^*$ such that B_p is the maximal integral manifold through p of the foliation of $\tilde{M}$ induced by M_1^*.

We shall show that the splitting $\tilde{M} = M_1^* \times M_2^*$ is invariant under both Γ and Z and moreover
(i) $p_1^*(Z) \subseteq C(M_1^*)$, where $p_1^* : Z \to I(M_1^*)$ is the projection homomorphism,
(ii) $L(N) \subseteq M_1^*(\infty)$.
We shall then show that a Γ and Z invariant splitting $\tilde{M} = M_1^* \times M_2^*$ satisfies case (2) of the lemma if it satisfies properties (i) and (ii) above and maximizes the dimension of M_1^* with respect to these properties.

Assertion (ii) follows from the definitions of B_p and M_1^*. The splitting $\tilde{M} = M_1^* \times M_2^*$ is invariant under both Γ and Z by Theorem A since both groups normalize N and hence leave $L(N)$ invariant. Therefore for every element φ in Γ or Z we may write $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i = p_i^*(\varphi) \in I(M_i^*)$ and p_i^* is the projection homomorphism into $I(M_i^*)$ for $i = 1, 2$. If $\varphi = (\varphi_1, \varphi_2) \in Z$, then d_{φ_1} is bounded

above on M_1^* since $d_\varphi \leq d_\varphi(p)$ on B_p for every point $p \in \tilde{M}$ by the discussion above. Hence $\varphi_1 \in C(M_1^*)$ by the discussion in (1.4), which proves assertion (i) above.

Now let $\tilde{M} = M_1^* \times M_2^*$ be a Riemannian splitting that is invariant under Γ and Z, satisfies (i) and (ii) above and maximizes the dimension of M_1^* with respect to these properties. We show that $p_2^*(N) = \{1\}$, which will complete the proof of the lemma. Let Γ_2, N_2 and Z_2 denote $p_2^*(\Gamma)$, $p_2^*(N)$ and $p_2^*(Z)$ respectively. By Proposition 1.6b Γ_2 satisfies the duality condition in M_2^*. Our assumption from above is that $Z \not\subseteq C(\tilde{M})$ and in view of (i) this implies that $Z_2 \neq \{1\}$ and $Z_2 \not\subseteq C(M_2^*)$.

We assume that $N_2 \neq \{1\}$ and obtain a contradiction. Since N_2 is normalized by Γ_2 and centralized by Z_2 we may apply the arguments above to Γ_2, N_2 and Z_2 and obtain a nontrivial Riemannian splitting $M_2^* = \tilde{M}_\alpha \times \tilde{M}_\beta$ that is invariant under Γ_2 and Z_2 and such that

(i)$'$ $p_\alpha(Z_2) \subseteq C(\tilde{M}_\alpha)$, where $p_\alpha : \Gamma_2 \to I(\tilde{M}_\alpha)$ denotes the projection homomorphism,

(ii)$'$ $L(N_2) \subseteq \tilde{M}_\alpha(\infty)$.

Now let $\tilde{M}_1 = M_1^* \times \tilde{M}_\alpha$ and $\tilde{M}_2 = \tilde{M}_\beta$. Then the splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ is invariant under Γ and Z and satisfies

(i) $p_1(Z) \subseteq C(\tilde{M}_1)$, where $p_1 : \Gamma \to I(\tilde{M}_1)$ denotes the projection homomorphism,

(ii) $L(N) \subseteq \tilde{M}_1(\infty)$.

Since $\dim \tilde{M}_1 > \dim M_1^*$ we obtain a contradiction to the hypothesis that M_1^* has maximal dimension with respect to Γ and Z invariant splittings of $\tilde{M}$ that satisfy (i) and (ii). This proves that $N_2 = p_2^*(N) = \{1\}$, which completes the proof of the lemma. □

Our next result is Theorem 2.4 of [**CE**].

THEOREM B. *Let $N \subseteq I(\tilde{M})$ be an abelian group whose normalizer Γ in $I(\tilde{M})$ satisfies the duality condition. Then $N \subseteq C(\tilde{M})$.*

PROOF. In the notation of the lemma note that $N \subseteq Z$ since N is abelian. In either of the two cases of the lemma it is now clear that $N \subseteq C(\tilde{M})$. □

The next result is a part of Theorem 4.2 of [**CE**].

THEOREM C. *Let $\Gamma \subset I(\tilde{M})$ satisfy the duality condition and suppose that the centralizer Z of Γ in $I(\tilde{M})$ is not the identity. Then there exists a Riemannian splitting $\tilde{M} = M_0^* \times M_1^*$ that is invariant under Γ and Z and such that*

(1) *M_0^* is a Euclidean space of positive dimension,*

(2) *$Z = C(M_0^*) \times \{1\}$,*

(3) *$p_0^*(\Gamma) \subseteq C(M_0^*)$,*

(4) *$p_1^*(\Gamma)$ has trivial centralizer in $I(M_1^*)$.*

Here p_0^*, p_1^* denote the projection homomorphisms into $I(M_0^*)$, $I(M_1^*)$ respectively.

PROOF. We let $N = \Gamma$ in the notation of the lemma above. Possibility 2(a) of the Lemma does not arise since Γ satisfies the duality condition and hence $Z \subseteq C(\tilde{M})$ by the Lemma. By the discussion in (1.4) we can write $\tilde{M}$ as a Riemannian product $M_0^* \times M_1^*$, where M_0^* is a Euclidean space of positive dimension, $L(Z) = M_0^*(\infty)$

and $\varphi = (\varphi_0, \mathrm{id})$ for every $\varphi \in Z$ with φ_0 a translation of M_0^*. It follows that $p_0^*(\Gamma) \subseteq C(M_0^*)$ since Γ commutes with Z, and hence $Z = C(M_0^*) \times \{1\}$, which proves (2) and (3). Finally (4) follows from (2). $\square$

The next result except for assertion (4) is Theorem 1 of [**S**].

THEOREM D. *Let $\Gamma \subseteq I(\tilde{M})$ satisfy the duality condition, and suppose that $\Gamma = A \cdot B$, where A and B are commuting subgroups of Γ. Then there exists a Γ-invariant Riemannian splitting $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \tilde{M}_2$ such that*
   (1) *$\tilde{M}_0$ is a Euclidean space of dimension $r \geq 0$ and $p_0(\Gamma) \subseteq C(\tilde{M}_0)$.*
   (2) *$C(\tilde{M}_0) \times \mathrm{id} \times \mathrm{id}$ is the centralizer Z of Γ in $I(\tilde{M})$.*
   (3) (a) *If $\varphi \in A$, then $\varphi = (\varphi_0, \varphi_1, \mathrm{id})$, where $\varphi_0 \in C(\tilde{M}_0)$ and $\varphi_1 \in I(\tilde{M}_1)$.*
       (b) *If $\varphi \in B$, then $\varphi = (\varphi_0, \mathrm{id}, \varphi_2)$, where $\varphi_0 \in C(\tilde{M}_0)$ and $\varphi_2 \in I(\tilde{M}_2)$.*
   (4) *If Γ is discrete, then $p_1(\Gamma) = p_1(A)$ and $p_2(\Gamma) = p_2(B)$ are discrete and have trivial centralizer in $I(\tilde{M}_1)$, $I(\tilde{M}_2)$ respectively.*
Again, $p_i : \Gamma \to I(\tilde{M}_i)$ denotes the projection homomorphism for $i = 0, 1, 2$. Any one of the factors in the results above may be absent and $\tilde{M} = \tilde{M}_0$ is also possible.

PROOF. We shall consider only the case that Z, the centralizer of Γ in $I(\tilde{M})$, is nontrivial. If $Z = \{1\}$, then the factor $\tilde{M}_0$ is absent in the final splitting of $\tilde{M}$ and we proceed as below. Assuming that $Z \neq \{1\}$ we use Theorem C to obtain a Γ-invariant splitting $\tilde{M} = M_0^* \times M_1^*$, where M_0^* is a Euclidean space of positive dimension and $Z = C(M_0^*) \times \{1\}$. If $p_i^* : \Gamma \to I(M_i^*)$ denotes the projection homomorphism for $i = 0, 1$, then clearly $p_0^*(\Gamma) \subseteq C(M_0^*)$. Let Γ_1, A_1, B_1 denote $p_1^*(\Gamma)$, $p_1^*(A)$, $p_1^*(B)$ respectively. Then $\Gamma_1 = A_1 \cdot B_1$, where A_1 and B_1 commute. By the characterization of Z the group Γ_1 has a trivial centralizer in M_1^*. We consider only the case that A_1 and B_1 are both nonidentity subgroups of $I(M_1^*)$. The remaining cases are handled similarly and correspond to the absence of one or both of the factors $\tilde{M}_1, \tilde{M}_2$ in the statement of the theorem. We now apply the lemma at the beginning of this section to the group $N = A_1$ and note that $1 \neq B_1 \subseteq Z_1$, where Z_1 denotes the centralizer of A_1 in $I(M_1^*)$. If $Z_1 \subseteq C(M_1^*)$, then B_1 is abelian and hence lies in the center of Γ_1, contradicting the fact that Γ_1 has trivial centralizer in $I(M_1^*)$. Hence from the lemma of this section we obtain a nontrivial Riemannian splitting $M_1^* = \tilde{M}_\alpha \times \tilde{M}_\beta$ that is invariant under Γ_1 and Z_1 and such that $A_1 \subseteq I(\tilde{M}_\alpha) \times \{1\}$ and $p_\alpha(Z_1) \subseteq C(\tilde{M}_\alpha)$, where p_α is the projection homomorphism into $I(\tilde{M}_\alpha)$. Since $p_\alpha(Z_1)$ is abelian and contains $p_\alpha(B_1)$ it follows that $p_\alpha(B_1)$ commutes with both A_1 and B_1 and hence with $\Gamma_1 = A_1 \cdot B_1$. Therefore $p_\alpha(B_1) = \{1\}$ since Γ_1 has trivial centralizer in $I(\tilde{M}_1)$. If we define $\tilde{M}_0 = M_0^*$, $\tilde{M}_1 = \tilde{M}_\alpha$ and $\tilde{M}_2 = \tilde{M}_\beta$, then the splitting $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \tilde{M}_2$ is invariant under Γ and satisfies assertions (1), (2) and (3) of Theorem D.

We prove assertion (4) of the theorem. We shall need the following.

LEMMA. *Let $N \subseteq I(\tilde{M})$ be a discrete group and let G denote the normalizer of N in $I(\tilde{M})$. Then G_0, the connected component of G that contains the identity, lies in the centralizer of N in $I(\tilde{M})$.*

PROOF OF THE LEMMA. Let $\varphi \in N$ be given. If $g(t)$ is a curve in G that starts at the identity, then $\varphi(t) = g(t)\varphi g(t)^{-1}$ is a curve in N that starts at φ. The discreteness of N implies that $\varphi(t) = \varphi$ for all t and hence $\exp(\mathfrak{G})$ centralizes

N, where $\mathfrak{G}$ is the Lie algebra of G and $\exp: \mathfrak{G} \to G$ is the exponential map. The result follows since $\exp(\mathfrak{G})$ contains a neighborhood of the identity in G and any neighborhood of the identity in G generates G_0. $\square$

Assume that the group Γ in the statement of the theorem is discrete. If $p_1(\Gamma) \subseteq I(\tilde{M}_1)$ is not discrete, then $\overline{(p_1(\Gamma))}_0$ centralizes Γ by the lemma just proved, which contradicts assertion (2) of the theorem. Therefore $p_1(\Gamma)$ is discrete in $I(\tilde{M}_1)$ and similarly $p_2(\Gamma)$ is discrete in $I(\tilde{M}_2)$. Finally it follows from assertion (3) of the theorem that $p_1(\Gamma) = p_1(A)$ and $p_2(\Gamma) = p_2(B)$. The fact that $p_1(\Gamma)$ has trivial centralizer in $I(\tilde{M}_i)$ for $i = 1, 2$ is a consequence of assertion (2). $\square$

Our next result is essentially the main theorem in §3 of [**E3**] but the proof here is shorter and conceptually simpler.

THEOREM E. *Let $\tilde{M}$ have a trivial Euclidean de Rham factor. Let $G \subseteq I(\tilde{M})$ be a closed, connected Lie subgroup of positive dimension whose normalizer Γ in $I(\tilde{M})$ satisfies the duality condition. Then there exists a Γ-invariant Riemannian splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ such that*
(1) *$\tilde{M}_1$ is a symmetric space of noncompact type.*
(2) *$G = I_0(\tilde{M}_1) \times \{1\}$.*

PROOF. We shall need three preliminary results.

LEMMA E1. *G is a semisimple Lie group with trivial center and no compact normal subgroups except the identity.*

LEMMA E2. *If $L(G) = \tilde{M}(\infty)$, then $\tilde{M}$ is a symmetric space of noncompact type and $G = I_0(\tilde{M})$.*

LEMMA E3. *Let $\tilde{M}$ be an arbitrary, complete, simply connected manifold of nonpositive sectional curvature. Let $G \subseteq I(\tilde{M})$ be a closed, connected Lie subgroup of positive dimension that is semisimple with finite center and has no compact normal subgroups except the identity. Let Γ denote the normalizer of G in $I(\tilde{M})$ and let $Z_\Gamma(G) = \{\varphi \in \Gamma: \varphi g = g\varphi \text{ for all } g \in G\}$. Then $\Gamma_0 = G \cdot Z_\Gamma(G)$ has finite index in Γ.*

PROOF OF THE LEMMAS. Lemma E1 is Lemma 3.1 of [**E3**] and the proof of Lemma E2 is contained in the proof of Lemma 3.4 of [**E3**]. We prove Lemma E3 in two steps.

Step 1. Let Γ, G be as in Lemma E3 and let K be a maximal compact subgroup of G. Let $N_\Gamma(K)$ denote the normalizer of K in Γ. Then
(1) $\Gamma = G \cdot N_\Gamma(K)$.
(2) If $\mathfrak{G}, \mathfrak{K}$ denote the Lie algebras of G, K then $\text{Aut}_{\mathfrak{K}}(\mathfrak{G}) = \{\varphi \in \text{Aut}(\mathfrak{G}): \varphi(\mathfrak{K}) \subseteq \mathfrak{K}\}$ is a compact subgroup of $\text{Aut}(\mathfrak{G})$.

REMARK. With regard to (2) we recall that $\text{Aut}(\mathfrak{G})$ denotes the group of Lie algebra automorphisms of $\mathfrak{G}$. Since $\text{Ad}(G)$ is a connected Lie subgroup of $\text{Aut}(\mathfrak{G})$ whose Lie algebra is $\text{ad}(\mathfrak{G}) = \{\text{ad}\,X: X \in \mathfrak{G}\}$ it follows from the conditions on G that $\text{Ad}(G)$ is the connected component of $\text{Aut}(\mathfrak{G})$ that contains the identity; see for example Corollary 6.5 of [**H**, p. 122].

PROOF OF STEP 1. If X denotes the coset space G/K, then X equipped with a suitable left invariant metric becomes a Riemannian symmetric space of noncompact type; see for example Proposition 3.6 of [**H**, p. 178] and Theorem 3.1

of [**H**, p. 205]. The group K has a fixed point p in X by the Cartan fixed point theorem, and in fact $K = \{\varphi \in G : \varphi(p) = p\}$ since the latter is a compact subgroup of $I(\tilde{M})$. Since G acts transitively on X all maximal compact subgroups of G are conjugate by elements of G.

We prove (1). Given an element $\varphi \in \Gamma$ the subgroup $\varphi K \varphi^{-1}$ is a maximal compact subgroup of $\varphi G \varphi^{-1} = G$ and hence $\varphi K \varphi^{-1} = gKg^{-1}$ for some $g \in G$ by the remarks above. If $\psi = g^{-1}\varphi$, then $\psi \in N_\Gamma(K)$ and hence $\varphi = g\psi \in G \cdot N_\Gamma(K)$. □

We prove (2). For a discussion of definitions and facts relevant to this proof see, for example, [**E7**, pp. 121–133]. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be the Cartan decomposition of G induced by $\mathfrak{K}$, where $\mathfrak{K}$ is the Lie algebra of K and $\mathfrak{P}$ is the orthogonal complement of $\mathfrak{K}$ in $\mathfrak{G}$ with respect to the Killing form B of $\mathfrak{G}$. It is known that B is positive definite on $\mathfrak{P}$ and negative definite on $\mathfrak{K}$. If $\varphi \in \mathrm{Aut}_\mathfrak{K}(\mathfrak{G})$, then φ preserves B [**H**, p. 121] and hence $\varphi(\mathfrak{P}) \subseteq \mathfrak{P}$ since $\varphi(k) \subseteq \mathfrak{K}$. It follows that φ preserves the positive definite inner product Φ on $\mathfrak{G}$ given by $\Phi(k, \mathfrak{P}) = 0$, $\Phi = B$ on $\mathfrak{P}$ and $\Phi = -B$ on $\mathfrak{K}$. This proves (2) since $\mathrm{Aut}_\mathfrak{K}(\mathfrak{G})$ is a closed subgroup of the orthogonal group $O(\mathfrak{G}, \Phi)$, which is compact.

Step 2. We prove that $N_\Gamma(K) \cap [G \cdot Z_\Gamma(G)]$ has finite index in $N_\Gamma(K)$. In combination with assertion (1) of Step 1 this will complete the proof of Lemma E3.

We begin by defining a homomorphism $\rho \colon \Gamma \to \mathrm{Aut}(\mathfrak{G})$ by setting

$$\varphi e^{tX} \varphi^{-1} = e^{t\rho(\varphi)X}$$

for all $\varphi \in \Gamma$, $X \in \mathfrak{G}$ and $t \in \mathbf{R}$. The left-hand side of the expression above is a 1-parameter subgroup of G for every $\varphi \in \Gamma$ and $X \in \mathfrak{G}$, and the subgroup lies in $\mathfrak{K}$ if $\varphi \in N_\Gamma(K)$ and $X \in \mathfrak{K}$. Hence we obtain by restriction a homomorphism

$$\rho \colon N_\Gamma(K) \to \mathrm{Aut}_\mathfrak{K}(\mathfrak{G}).$$

Now let N_0 denote $N_\Gamma(K) \cap [G \cdot Z_\Gamma(G)]$ and suppose that the assertion of Step 2 is false. Then N_0 has infinite index in $N_\Gamma(K)$ and we may choose a sequence $\{\varphi_k\} \subseteq N_\Gamma(K)$ so that the cosets $\{\varphi_k N_0\}$ are all distinct. By the remark preceding the proof of Step 1 we can find a neighborhood $0 \subseteq \mathrm{Aut}(\mathfrak{G})$ of the identity such that $0 \subseteq \mathrm{Ad}(G)$. Since $\{\rho(\varphi_k)\}$ is a bounded sequence in $\mathrm{Aut}_\mathfrak{K}(\mathfrak{G})$ by assertion (2) of Step 1, we can find distinct integers n and m such that $\rho(\varphi_n^{-1}\varphi_m) \in 0$. Choose $g \in G$ so that $\rho(\varphi_n^{-1}\varphi_m) = \mathrm{Ad}(g)$, and let $\xi = g^{-1}\varphi_n^{-1}\varphi_m$. From the definition of ρ and Ad it follows that $\xi e^{tX} \xi^{-1} = e^{tX}$ for all $X \in \mathfrak{G}$ and all $t \in \mathbf{R}$. Hence $\xi \in Z_\Gamma G$ since $\exp(\mathfrak{G})$ contains a neighborhood $0^* \subseteq G$ of the identity and 0^* generates G. Therefore $\varphi_n^{-1}\varphi_m = g\xi \in N_0$ and it follows that $\varphi_n N_0 = \varphi_m N_0$, contradicting the way in which the sequence $\{\varphi_k\}$ was chosen. Therefore N_0 has finite index in $N_\Gamma(K)$, which completes the proof of Lemma E3 as indicated above. □

PROOF OF THEOREM E. Let Γ, G be as in the statement of the theorem, and let $\Gamma_0 = G \cdot Z_\Gamma(G)$. By Lemma E3 Γ_0 has finite index in Γ and hence Γ_0 satisfies the duality condition by Proposition 1.6b. By hypothesis $\tilde{M}$ has no Euclidean de Rham factor and hence it follows from Theorem D that there exists a Γ_0-invariant Riemannian splitting $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ such that $G \subseteq I(\tilde{M}_1) \times \{1\}$ and $Z_\Gamma(G) \subseteq \{1\} \times I(\tilde{M}_2)$. If $p_1 \colon \Gamma_0 \to I(\tilde{M}_1)$ denotes the projection homomorphism, then $G = p_1(\Gamma_0)$ and hence G satisfies the duality condition in $\tilde{M}_1$ by Proposition 1.6b.

Regarding G as a subgroup of $I(\tilde{M}_1)$ and applying Lemma E2 we conclude that $\tilde{M}_1$ is a symmetric space of noncompact type and $G = I_0(\tilde{M}_1)$. □

As a corollary of the result just proved we obtain

THEOREM F. *Let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \cdots \times \tilde{M}_k$ be the de Rham decomposition of $\tilde{M}$, where $\tilde{M}_0$ is the Euclidean factor of $\tilde{M}$ and $\{\tilde{M}_i : 1 \leq i \leq k\}$ are the irreducible non-Euclidean factors of $\tilde{M}$. Let $\Gamma \subseteq I(\tilde{M})$ be a subgroup that leaves the splitting invariant and satisfies the duality condition. Let $p_i : \Gamma \to I(\tilde{M}_i)$ denote the projection homomorphism for $0 \leq i \leq k$. If N is a nonidentity normal subgroup of Γ, then for every integer $i \geq 1$ we have either*

(1) *$p_i(N)$ is a discrete subgroup of $I(\tilde{M}_i)$ or*

(2) *$\tilde{M}_i$ is a symmetric space of noncompact type and $I_0(\tilde{M}_i) \subseteq \overline{p_i(N)}$.*

Moreover, if N is discrete, then $p_i(\Gamma)$ is discrete for some integer $i \geq 1$ if $p_i(N) \neq \{1\}$.

PROOF. We consider first the case that N is an arbitrary nonidentity normal subgroup of Γ, not necessarily discrete. Suppose that $p_i(N)$ is not discrete for some $i \geq 1$ and let $G = \overline{(p_i(N))}_0$. Then G is a connected Lie subgroup of $I(\tilde{M}_i)$ of positive dimension and G is normalized by $p_i(\Gamma)$, which satisfies the duality condition by Proposition 1.6b. Since $\tilde{M}_i$ is irreducible it follows from Theorem E that $\tilde{M}_i$ is a symmetric space of noncompact type and $G = I_0(\tilde{M}_i)$.

Now suppose that N is a discrete nonidentity normal subgroup of Γ and $p_i(N) \neq \{1\}$ for some $i \geq 1$. Suppose that $p_i(\Gamma)$ is not discrete and let $G = \overline{(p_i(\Gamma))}_0$. The group G is a connected Lie subgroup of $I(\tilde{M}_i)$ of positive dimension that normalizes $p_i(N)$ in $I(\tilde{M}_i)$ and hence normalizes N in $I(\tilde{M})$. By the lemma used to prove assertion (4) of Theorem D it follows that G centralizes N in $I(\tilde{M})$ and hence centralizes $p_i(N)$ in $I(\tilde{M}_i)$. This contradicts the corollary to the lemma at the beginning of this section since $\tilde{M}_i$ is irreducible. Therefore $p_i(\Gamma)$ is discrete. □

The next result is Lemma A of [**E1**].

COROLLARY F. *Let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1$, where $\tilde{M}_0$ is the Euclidean de Rham factor of $\tilde{M}$ and $\tilde{M}_1$ is the product of all non-Euclidean de Rham factors of $\tilde{M}$. Let $p_i : I(\tilde{M}) \to I(\tilde{M}_i)$ denote the projection homomorphism for $i = 0, 1$. If $\Gamma \subseteq I(\tilde{M})$ is a discrete group that satisfies the duality condition, then $p_1(\Gamma)$ is a discrete subgroup of $I(\tilde{M}_1)$.*

PROOF. Let $M = M_0^* \times M_1^* \times \cdots \times M_k^*$ denote the de Rham decomposition of $\tilde{M}$, where $M_0^* = \tilde{M}_0$ and $M_1^* \times \cdots \times M_k^*$ is the de Rham decomposition of $\tilde{M}_1$. Let Γ^* denote the subgroup of Γ that leaves invariant the de Rham decomposition. Then Γ^* has finite index in Γ since the elements of $I(\tilde{M})$ leave invariant the foliation of $\tilde{M}$ induced by M_0^* and permute the foliations induced by M_i^* for $i \geq 1$. Let $p_i^* : \Gamma^* \to I(M_i^*)$ denote the projection homomorphism for $0 \leq i \leq k$. By Proposition 1.6b it follows that Γ^* satisfies the duality condition in $\tilde{M}$ and $p_i^*(\Gamma^*)$ satisfies the duality condition in M_i^* for every i. Setting $N = \Gamma^*$ and applying the previous result to Γ^* we find that $p_i^*(\Gamma^*)$ is discrete for every $i \geq 1$. Hence $p_1(\Gamma^*)$ is a discrete subgroup of $I(\tilde{M}_1)$ since $p_1(\Gamma^*)$ is a subgroup of $p_1^*(\Gamma^*) \times \cdots \times p_k^*(\Gamma^*)$. We conclude that $p_1(\Gamma)$ is a discrete subgroup of $I(\tilde{M}_1)$ since $p_1(\Gamma^*)$ has finite index in $p_1(\Gamma)$. □

5. Lattices. Let $\tilde{M}$ be any complete, simply connected manifold with sectional curvature $K \le 0$.

DEFINITIONS. If $\Gamma \subseteq I(\tilde{M})$ is a discrete subgroup, then $\operatorname{vol}(\tilde{M}/\Gamma) = \inf\{A > 0: \operatorname{vol}(O) \le A$ for any open set $O \subseteq \tilde{M}$ such that $\varphi(O) \cap O$ is empty for all $\varphi \ne 1$ in $\Gamma\}$. A discrete group Γ is called a *lattice* if $\operatorname{vol}(\tilde{M}/\Gamma)$ is finite. A lattice Γ is *uniform* (*nonuniform*) if the quotient space $\tilde{M}/\Gamma$ is compact (noncompact).

If $\Gamma \subseteq I(\tilde{M})$ is a discrete group, then the set of points in $\tilde{M}$ fixed by some nonidentity element of Γ is a closed nowhere dense subset of $\tilde{M}$. If $p \in \tilde{M}$ is a point not fixed by any nonidentity element of Γ, then we define the *Dirichlet fundamental domain* R_p with center p:

$$R_p = \{q \in \tilde{M}: d(p,q) < d(p, \varphi q) \text{ for all } \varphi \ne 1 \text{ in } \Gamma\}.$$

PROPOSITION G. *Let $\Gamma \subseteq I(\tilde{M})$ be a discrete group and let p be a point of $\tilde{M}$ that is not fixed by any nonidentity element of Γ. Then*

(1) *R_p is an open subset of $\tilde{M}$ that contains p.*
(2) *$\varphi(R_p) \cap R_p$ is empty for all $\varphi \ne 1$ in Γ.*
(3) *$\tilde{M} = \bigcup_{\varphi \in \Gamma} \varphi(\overline{R_p})$,*
(4) *$\operatorname{vol}(R_p) = \operatorname{vol}(\tilde{M}/\Gamma)$.*

We omit the proof, which is routine.

PROPOSITION H. *Let $\Gamma \subseteq I(\tilde{M})$ be a lattice, and let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ be a Riemannian splitting that is invariant under Γ. Let $p_i: \Gamma \to I(\tilde{M}_i)$ be the projection homomorphism for $i = 1, 2$. If $p_2(\Gamma)$ is a discrete subgroup of $I(\tilde{M}_2)$, then $p_2(\Gamma)$ is a lattice in $I(\tilde{M}_2)$ and $\operatorname{kernel}(p_2)$ is a lattice in $I(\tilde{M}_1)$.*

REMARK. This result not only strengthens the conclusion of assertion (1) in Theorem 4.1 of [**E5**] but it also considerably simplifies the proof.

PROOF. Let N denote the kernel of p_2. Clearly N is a discrete subgroup of $I(\tilde{M}_1)$. Let $O_1 \subseteq \tilde{M}_1$ be any open set such that $\varphi(O_1) \cap O_1$ is empty for all $\varphi \ne 1$ in N. Let $O_2 \subseteq \tilde{M}_2$ be any open set such that $\varphi_2(O_2) \cap O_2$ is empty for all $\varphi_2 \ne 1$ in $p_2(\Gamma)$; such open sets exist by the discreteness of $p_2(\Gamma)$. If $O = O_1 \times O_2$, then O is open in $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ and it follows from the definition of O_1 and O_2 that $\varphi(O) \cap O$ is empty for all $\varphi \ne 1$ in Γ. Since $\Gamma \subseteq I(\tilde{M})$ is a lattice there exists a number $A > 0$ depending only on Γ such that $\operatorname{vol}(O_1) \cdot \operatorname{vol}(O_2) = \operatorname{vol}(O) \le A$. Hence $\operatorname{vol}(O_1) \le A/\operatorname{vol}(O_2)$. By the arbitrary choice of the open set O_1 it follows that $N = \operatorname{kernel}(p_2)$ is a lattice in $I(\tilde{M}_1)$. Similarly $\operatorname{vol}(O_2) \le A/\operatorname{vol}(O_1)$ and by the arbitrary choice of the open set O_2 it follows that $p_2(\Gamma)$ is a lattice in $I(\tilde{M}_2)$. □

The next result is a restatement of the main theorem of [**E1**], but the proof here simplifies greatly the proof of [**E1**]. The next result is also a special case of Proposition 2.4 and Theorem 4 of [**E4**], but the extra generality in [**E4**] at present requires a much more complicated proof.

THEOREM I. *Let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1$, where $\tilde{M}_0$ is the Euclidean de Rham factor of $\tilde{M}$ and $\tilde{M}_1$ is the product of all non-Euclidean de Rham factors of $\tilde{M}$. Let*

$p_i \colon I(\tilde{M}) \to I(\tilde{M}_i)$ *denote the projection homomorphism for* $i = 0, 1$. *Let* $\Gamma \subseteq I(\tilde{M})$ *be a lattice. Then*

(1) $\mathrm{kernel}(p_1)$ *is a uniform lattice in* $\tilde{M}_0$.

(2) *The dimension of* $\tilde{M}_0$ *equals the rank of the Clifford subgroup of* Γ, *the unique maximal normal abelian subgroup of* Γ.

PROOF. By Corollary F of the previous section the group $p_1(\Gamma)$ is discrete in $I(\tilde{M}_1)$, and by Proposition H above we conclude that $\mathrm{kernel}(p_1)$ is a lattice in $\tilde{M}_0$. A theorem of Bieberbach [**W2**, p. 100] says that $C_1 = \mathrm{kernel}(p_1) \cap C(\tilde{M}_0)$ is a normal abelian subgroup of finite index in $\mathrm{kernel}(p_1)$. Hence both C_1 and $\mathrm{kernel}(p_1)$ are uniform lattices in $I(\tilde{M}_0)$, which proves (1).

If $C(\Gamma) = \Gamma \cap C(\tilde{M})$, then $C(\Gamma)$ is a normal abelian subgroup of Γ by the discussion in (1.4), and $C(\Gamma)$ is the unique maximal abelian normal subgroup of Γ by Theorem B of §4. The group $C_1 \subseteq C(\tilde{M}_0)$ defined above is a uniform lattice in $I(\tilde{M}_0)$, and hence $\dim \tilde{M}_0 = \mathrm{rank}\, C_1 \leq \mathrm{rank}\, C(\Gamma)$ since $C_1 \subseteq C(\Gamma)$. Conversely, $C(\Gamma)$ is a discrete subgroup of $C(\tilde{M}_0) \times \{1\}$, and hence $\mathrm{rank}\, C(\Gamma) \leq \dim \tilde{M}_0$, which completes the proof of (2) and of the theorem. □

We conclude with two results on lattices that are nontrivial direct products of subgroups. The first of these results is Theorem 2 of [**S**]. It was proved in the compact C^∞ case by Gromoll and Wolf [**GW**] and in the compact real analytic case by Lawson and Yau [**LY**]. The second result is similar to Proposition 2.6 of [**E4**].

THEOREM J. *Let* $\Gamma \subseteq I(\tilde{M})$ *be a lattice with trivial center that splits as a direct product* $\Gamma_1 \times \Gamma_2$ *of subgroups* Γ_1 *and* Γ_2. *Then there exists a Riemannian splitting* $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ *that is invariant under* Γ *and such that* $\Gamma_1 \subseteq I(\tilde{M}_1) \times \{1\}$ *and* $\Gamma_2 \subseteq \{1\} \times I(\tilde{M}_2)$. *In particular* Γ_i *is a lattice in* $I(\tilde{M}_i)$ *for* $i = 1, 2$.

PROOF. Let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \tilde{M}_2$ be the Γ-invariant splitting of $\tilde{M}$ that has the properties of the statement of Theorem D of §4, where we set $A = \Gamma_1$ and $B = \Gamma_2$ in the notation of that result. By Theorem D it suffices to show that $\tilde{M}_0$ has dimension zero. Suppose that $\tilde{M}_0$ has positive dimension and let M_1^* denote $\tilde{M}_1 \times \tilde{M}_2$, with $p_1^* \colon \Gamma \to I(M_1^*)$ denoting the corresponding projection homomorphism. By assertion (4) of Theorem D the group $p_1(\Gamma) \times p_2(\Gamma)$ is discrete in $I(M_1^*)$ and hence the subgroup $p_1^*(\Gamma)$ is also discrete. By Proposition H we see that $\mathrm{kernel}(p_1^*)$ is a lattice in $I(\tilde{M}_0)$ and in particular is not the identity. By assertion (1) of Theorem D we see that $\mathrm{kernel}(p_1^*) \subseteq C(\tilde{M}_0)$ and hence $\mathrm{kernel}(p_1^*)$ lies in the center of Γ by assertion (2) of Theorem D. This contradicts the hypothesis that Γ has trivial center and completes the proof of Theorem J. □

THEOREM K. *Let* $\Gamma \subseteq I(\tilde{M})$ *be a lattice that splits as a direct product* $\Gamma_1 \times \Gamma_2$ *of subgroups* Γ_1 *and* Γ_2. *Then for* $i = 1, 2$ *the group* Γ_i *is isomorphic to a lattice in a Riemannian factor of* $\tilde{M}$.

REMARK. It is not necessarily true that each group Γ_i is itself a lattice in some Riemannian factor. See for example the proof of Proposition 3.2 of [**E4**].

PROOF OF THE THEOREM. For later convenience of notation we set $A = \Gamma_1$ and $B = \Gamma_2$ so that Γ is the direct product $A \cdot B$. Let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \tilde{M}_2$ be the Γ-invariant splitting of Theorem D in the previous section. We may assume that $\tilde{M}_0$

has positive dimension for otherwise we are done by Theorem D. Let A_0, B_0 and Γ_0 denote respectively the centers of A, B and Γ. In particular $A_0 \cap B_0 = \{1\}$ and Γ_0 is the direct product $A_0 \cdot B_0$. By (4) of Theorem D the projection $(p_1 \times p_2)(\Gamma)$ is discrete in $I(\tilde{M}_1 \times \tilde{M}_2)$ and it now follows from (1) of Theorem D and Proposition H that $\Gamma_0 = \text{kernel}(p_1 \times p_2)$ is a uniform lattice in $I(\tilde{M}_0)$.

It suffices to prove the theorem for the subgroup A. We prove the result in three steps.

Step 1. $\Gamma^* = \text{kernel}(p_2)$ is a lattice in $I(\tilde{M}_0 \times \tilde{M}_1)$ and $\Gamma^* = A \cdot B_0$.

PROOF. By (4) of Theorem D and Proposition H it follows that $\Gamma^* = \text{kernel}(p_2)$ is a lattice in $I(\tilde{M}_0 \times \tilde{M}_1)$. Clearly $A \cdot B_0 \subseteq \Gamma^*$ by assertion (3) of Theorem D and the fact that $B_0 \subseteq \Gamma_0 = \text{kernel}(p_1 \times p_2)$. Conversely let $\varphi \in \Gamma^* \subseteq \Gamma$ be given and write $\varphi = ab$ for suitable elements $a \in A$ and $b \in B$. Then $b = a^{-1}\varphi \in \Gamma^*$ by assertion (3a) of Theorem D, and it follows that $b \in C(\tilde{M}_0)$ by assertion (3b) of Theorem D. Therefore $b \in B_0$ and $\varphi \in A \cdot B_0$ by assertion (2) of Theorem D. This completes the proof of Step 1.

Before stating Steps 2 and 3 we need some preliminary discussion. We may assume that $B_0 \neq \{1\}$ for otherwise $\Gamma^* = A$ is a lattice in $I(\tilde{M}_0 \times \tilde{M}_1)$ by Step 1. By assertion (1) of Theorem D the group B_0 (and in fact $p_0(\Gamma)$) consists of translations of the Euclidean space $\tilde{M}_0$. Hence we may write

$$\tilde{M}_0 = \tilde{M}_\alpha \times \tilde{M}_\beta, \quad \text{Riemannian product,}$$

such that $B_0 \subseteq \{1\} \times I(\tilde{M}_\beta)$ and B_0 is a uniform lattice of $I(\tilde{M}_\beta)$. This splitting of $\tilde{M}_0$ is invariant under the abelian group $p_0(\Gamma)$ since B_0 is a normal subgroup of $p_0(\Gamma)$. Hence the splitting $\tilde{M}_0 \times \tilde{M}_1 = \tilde{M}_\alpha \times \tilde{M}_\beta \times \tilde{M}_1$ of $\tilde{M}_0 \times \tilde{M}_1$ is invariant under $\Gamma^* = \text{kernel}(p_2)$. Let p_α, p_β and p_1 denote the projection homomorphisms of Γ^* into $I(\tilde{M}_\alpha)$, $I(\tilde{M}_\beta)$ and $I(\tilde{M}_1)$ respectively.

Step 2. $p_\alpha \times p_1 : A \to (p_\alpha \times p_1)(A) = (p_\alpha \times p_1)(\Gamma^*)$ is an isomorphism.

Step 3. $(p_\alpha \times p_1)(\Gamma^*)$ is discrete in $I(\tilde{M}_\alpha \times \tilde{M}_1)$.

These two steps will complete the proof of the theorem since $(p_\alpha \times p_1)(\Gamma^*)$ will then be a lattice in $I(\tilde{M}_\alpha \times \tilde{M}_1)$ by Proposition H together with the fact that Γ^* is a lattice in $I(\tilde{M}_0 \times \tilde{M}_1)$.

PROOF OF STEP 2. Observe that $(p_\alpha \times p_1)(A) = (p_\alpha \times p_1)(\Gamma^*)$ by the definition of $\tilde{M}_\beta$ and the fact that $\Gamma^* = A \cdot B_0$. To complete the proof of Step 2 it suffices to show that $A \cap \text{kernel}(p_\alpha \times p_1) = \{1\}$. Let $\varphi \in A$ be an element such that $p_\alpha(\varphi) = p_1(\varphi) = \{1\}$. By assertion (3) of Theorem D it follows that $\varphi = (\text{id}, \varphi_\beta, \text{id}) \in I(\tilde{M}_\alpha \times \tilde{M}_\beta \times \tilde{M}_1)$, where φ_β is a translation on $\tilde{M}_\beta$. The group Γ' generated by φ and B_0 is a discrete subgroup of translations of $\tilde{M}_\beta$ since $\Gamma' \subseteq \Gamma$ and Γ is discrete. Since B_0 is a uniform lattice in $I(\tilde{M}_\beta)$ it is possible for Γ' to be discrete only if B_0 has finite index in Γ', which means that $\varphi^n \in B_0$ for some positive integer n. Since $\varphi^n \in A \cap B_0 = \{1\}$ and φ is a translation of $\tilde{M}_\beta$ it follows that $\varphi = \{1\}$.

PROOF OF STEP 3. Let $\{\varphi_n\} \subseteq A$ be a sequence such that $(p_\alpha \times p_1)(\varphi_n) \to 1$ as $n \to +\infty$. To prove that $(p_\alpha \times p_1)(\Gamma^*) = (p_\alpha \times p_1)(A)$ is discrete in $I(\tilde{M}_\alpha \times \tilde{M}_1)$ it suffices to show that $p_\alpha(\varphi_n) = p_1(\varphi_n) = \{1\}$ for sufficiently large n. By hypothesis $p_1(\varphi_n) \to 1$ as $n \to \infty$ and hence $p_1(\varphi_n) = \{1\}$ for sufficiently large n since $p_1(\Gamma)$

is discrete in $I(\tilde{M}_1)$ by assertion (4) of Theorem D. Since B_0 is a uniform lattice in $I(\tilde{M}_\beta)$ we may choose a sequence $\{\beta_n\} \subseteq B_0$ so that $p_\beta(\varphi_n \beta_n)$ converges to some element β^* of $I(\tilde{M}_\beta)$ as $n \to \infty$. The elements $\psi_n = \varphi_n \beta_n$ lie in $\Gamma^* = A \cdot B_0$ and since $B_0 \subseteq \mathrm{kernel}(p_\alpha) \cap \mathrm{kernel}(p_1)$ we see that

$$p_\alpha(\psi_n) = p_\alpha(\varphi_n) \to 1 \quad \text{as } n \to +\infty,$$
$$p_\beta(\psi_n) \to \beta^* \quad \text{as } n \to +\infty,$$
$$p_1(\psi_n) = p_1(\varphi_n) = 1 \quad \text{for large } n.$$

Hence $\psi_n \to (\mathrm{id}, \beta^*, \mathrm{id})$ as $n \to \infty$. Since Γ^* is discrete we conclude that $\psi_n = \beta^*$ and $p_\alpha(\psi_n) = 1$ for sufficiently large n. This completes the proof of Step 3 and of the theorem. $\quad\square$

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